

Econ 802

Lecture Notes on Chapter 4

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This chapter addresses the topic of cost minimization. Two main reasons why this is important:

- ① Cost functions are very useful in thinking about the supply function of a competitive firm.
- ② Even if a firm is not perfectly competitive in its output market (maybe it has a monopoly), the firm still wants to minimize the cost of whatever output it produces. Regardless of how output is chosen, cost min is a necessary condition for profit max.

Consider the following problem:

$$\min w \cdot x \\ x \in Q(y)$$

where $w = (w_1, \dots, w_n) > 0$ is a vector of input prices and $x = (x_1, \dots, x_n) \geq 0$ is a vector of input quantities.

y is a scalar output. We could use the constraint $x \in V(y)$ but if $x \notin Q(y)$ we will be wasting some inputs, so may as well use $x \in Q(y)$. We can also write the constraint using the production function $y = f(x)$.

Notes: This is like a restricted profit function where some inputs are fixed while other inputs and output can be varied. Here, output is the thing that is fixed.

(2)

Let's assume differentiability ignore non-negativity constraints, and set up a Lagrangean:

$$L(x, d) = w \cdot x - d [f(x) - y]$$

$$\begin{aligned} \text{The FOC are } w &= d \frac{\partial f(x^*)}{\partial x} & \left. \vphantom{\frac{\partial f(x^*)}{\partial x}} \right\} n \text{ equations} \\ f(x^*) &= y & \left. \vphantom{f(x^*)} \right\} 1 \text{ equation} \end{aligned}$$

Ideally we would like to solve this system of $n+1$ equations for $x(w, y)$ and $d(w, y)$. If x^* really does solve the problem the solution for $x(w, y)$ will give the conditional input demand functions (because they are conditional on the output y).

$$\begin{aligned} \text{Note: from FOCs we have } w_i &= d \frac{\partial f(x^*)}{\partial x_i} \\ w_j &= d \frac{\partial f(x^*)}{\partial x_j} \text{ for all } i, j \end{aligned}$$

Divide the top equation by the bottom equation to get

$$\frac{w_i}{w_j} = \frac{\frac{\partial f(x^*)}{\partial x_i}}{\frac{\partial f(x^*)}{\partial x_j}} \text{ for all } i, j \Rightarrow \begin{aligned} &\text{The technical rate of substitution} \\ &= \text{price ratio.} \end{aligned}$$

Since the FOCs are necessary this must be true at any solution to the cost min problem. Later we will see that this implies a tangency between an isoquant and an isocost line.

(3)

Now Think about SOC.

The necessary SOC is $h \frac{\partial^2 f(x^*)}{\partial x^2} h \leq 0$

for all h satisfying $\frac{\partial f(x^*)}{\partial x} \cdot h = 0$

Two points: (1) You might be thinking that since this is a minimization problem, the SOC should involve pos semi-definiteness, not neg semi-definiteness. However, the requirement is that the ^{Hessian of the} Lagrangian be pos semidef. Because $\lambda > 0$ and there is a minus sign, this implies that the Hessian of the production function has to be negative semi-def.

(2) Why are we imposing the restriction on h ? This comes from the constraint in the cost min problem. Think about h as a small deviation from x^* . In order for the constraint to be satisfied we must have $f(x^* + h) = y$. If you expand $f(x^* + h)$ around x^* using a Taylor series you will find that for small h , $\frac{\partial f(x^*)}{\partial x} \cdot h \neq 0$ implies that output either goes up or down. Either way we violate the constraint. So we are limited to choices of h that yield zero. (This means we are moving along the isoquant for y)

From the FOC, we have $w = \lambda \frac{\partial f(x^*)}{\partial x}$ so we can write $\frac{\partial f(x^*)}{\partial x} \cdot h = 0$ as $w \cdot h = 0$. Sometimes you will see the SOC written this way - it means the same thing.

The restriction $\frac{\partial f(x^*)}{\partial x} \cdot h = 0$ leads to the use of bordered Hessians. I will come back to this later.

(4)

The SOC on p. 3 was necessary not sufficient.
The sufficient condition is

$$h \frac{\partial^2 f(x^*)}{\partial x^2} h < 0 \text{ for all } h \neq 0 \text{ with } \frac{\partial f(x^*)}{\partial x} \cdot h = 0$$

If $V(y)$ is convex (equivalently f is quasi-concave)
Then the necessary SOC will hold automatically.
Furthermore x^* will actually solve the cost min
problem. However, it may not be a unique solution.

If $V(y)$ is strictly convex (equivalently f is strictly
quasi-concave) Then x^* is the unique solution
[but note that strictness here is not quite enough
to guarantee that the sufficient SOC holds]

For more on concavity, convexity, SOC's, etc, see the
notes about this on my web site.

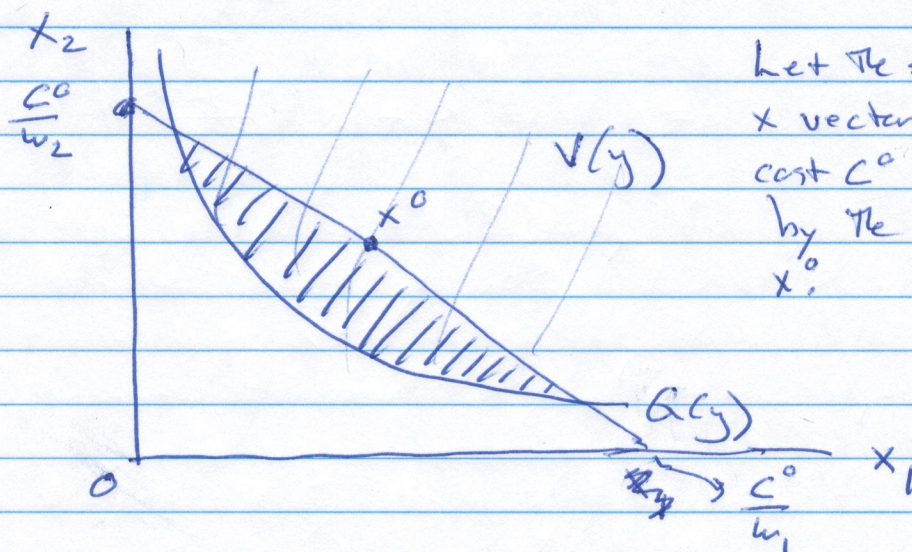
A deeper issue is: how do we know that a solution
even exists? This was a serious question for the
profit max problem, and you might be worried
that a similar question would arise here.

Luckily, you don't have to worry. There is a general
argument showing that some solution must exist.

Remember that we assumed in Ch. 1 that the set
 $V(y)$ is closed and non-empty for any $y \geq 0$. The
fact that it is non-empty means there is some input
vector that can produce y . Call this x^0 . There is
also a cost associated with x^0 , call it $C^0 = w x^0$.

(5)

Now draw a graph for the 2-input case.



Let the set of all x vectors with Total cost C^0 be given by the line through x^0 .

The equation for this isocost line is $C^0 = w_1 x_1 + w_2 x_2$ or $x_2 = \frac{C^0 - w_1 x_1}{w_2}$ so the slope is $-\frac{w_1}{w_2} < 0$.

Clearly no point above the isocost line through x^0 could be a solution to the cost min problem, because it would have a higher cost than x^0 . So we can limit our attention to the heavy shaded area between the isocost line and the isofunct (including the boundaries).

Because $V(y)$ is closed, the heavy shaded area is also a closed set. Also the objective function $w \cdot x$ is clearly a continuous function of x . There is a mathematical result called the Weierstrasse Theorem that says: any continuous function defined on a compact set takes on a maximum and a minimum value somewhere in the set. A set is compact if it is closed and bounded. The heavy shaded area satisfies both of these requirements, so $w \cdot x$ has a minimum value for some x in the set (although this x need not be unique). Thus the cost min problem has a solution.

(6)

Let's define the cost function as

$$c(w, y) = \min w \cdot x \text{ subject to } f(x) = y$$

so this is the minimum possible cost for the given w and y .

We can now think about the firm's profit max problem for the choice of y , which is

$$\max_{y \geq 0} \{py - c(w, y)\} \quad \text{If we ignore the non-negativity constraint, this yields}$$

$$\text{FOC: } p - \frac{\partial c(w, y)}{\partial y} = 0 \quad \text{or: price = marginal cost!}$$

The necessary SOC is $\frac{\partial^2 c}{\partial y^2} \geq 0$, which says marginal cost cannot be decreasing at a solution y^* .

The sufficient SOC is $\frac{\partial^2 c}{\partial y^2} > 0$, which says MC is rising at a solution y^* .

We'll come back to these ideas later in the course when we think about supply curves.

Note: you should carefully read Varian's derivations of cost functions for special cases like Cobb-Douglas, CES, Leontief, and linear production functions.

Use graphs to work out the shapes of isoquants, their slopes, the convexity or strict convexity of $V(y)$, and the nature of the cost-minimizing solution(s).

Also read the material on the algebraic approach to doing comparative statics.

The next topic I want to discuss is how to use the FOCs to do comparative statics on the Cost min problem.

Let's go back to the Lagrangean $L = wx - d[f(x) - y]$ and the FOCs

$$\lambda \frac{\partial f(x)}{\partial x} = w$$

$$f(x) = y$$

Assume we can use the implicit function Theorem to treat d and x as functions of (w, y) . This gives $x(w, y)$ and $d(w, y)$. If we substitute these back into the FOCs, we obtain the identities

$$d(w, y) \frac{\partial f[x(w, y)]}{\partial x} = w$$

$$f[x(w, y)] = y$$

In what follows, I will write $x(w)$ and $d(w)$ in place of $x(w, y)$ and $d(w, y)$ because I will hold y constant.

Since these are identities, we can differentiate on both sides with respect to the price vector w . This gives

$$d \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial w} + \frac{\partial f}{\partial x} \frac{\partial d}{\partial w} = I$$

$\begin{matrix} n \times n & n \times n & n \times 1 & 1 \times n & n \times n \end{matrix}$

$$\text{and } \frac{\partial f}{\partial x} \frac{\partial x}{\partial w} = 0$$

$\begin{matrix} 1 \times n & n \times n & 1 \times n \end{matrix}$

(8)

Let's rewrite This more compactly as

$$\begin{bmatrix} d \frac{\partial^2 f}{\partial x^2} & \frac{\partial f'}{\partial x} \\ \frac{\partial f}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial w} \\ \frac{\partial d}{\partial w} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

if we can assume non-singularity, This yields

$$\begin{bmatrix} \frac{\partial x}{\partial w} \\ \frac{\partial d}{\partial w} \end{bmatrix} = \begin{bmatrix} d \frac{\partial^2 f}{\partial x^2} & \frac{\partial f'}{\partial x} \\ \frac{\partial f}{\partial x} & 0 \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

where $\frac{\partial x}{\partial w}$ is the substitution matrix for the conditional input demand functions.

A few comments on This.

First, we are inverting a bordered Hessian matrix, where the borders are given by $\frac{\partial f}{\partial x}$ and $\frac{\partial f'}{\partial x}$.

Second, you will sometimes see bordered Hessians where the borders involve w rather than $\frac{\partial f}{\partial x}$. That's OK because the two vectors are related through the FOC (see bottom of p. 3).

Can we assume non-singularity? Actually we already did when we used the implicit function Theorem on p. 7.

All of This is fine if the sufficient SOC holds (see p. 4). In This case, we can use the implicit function Theorem, the conditional demand functions are well-defined and differentiable, the bordered Hessian is non-singular, etc.

(9)

When we have non-singularity, two things are true:

- ① Because the bordered Hessian is symmetric, the substitution matrix $\frac{\partial x}{\partial w}$ is also symmetric.
- ② We can also show that $\frac{\partial x}{\partial w}$ is negative definite (but this is hard with bordered Hessians; we can get a similar result much more easily using the duality approach - see chapter 5 (Shephard's Lemma)).

These results should not be too surprising given what we know from profit max, but it is nice to confirm that we get similar results for unconditional and conditional input demand functions (for example, input demand curves slope down).

The Elasticity of Substitution

Now we are ready to say more about the elasticity of substitution and CES production functions.

CES stands for "constant elasticity of substitution." What is constant? The degree to which one input substitutes for another, holding output fixed, when the price of an input changes.

Example: capital/labor substitution. If the wage rate goes up, in the long run firms may switch to more capital-intensive production techniques. But is this effect large or small? What happens to the share of total cost going to labor? Does it increase or decrease?

Let's define the elasticity of substitution as

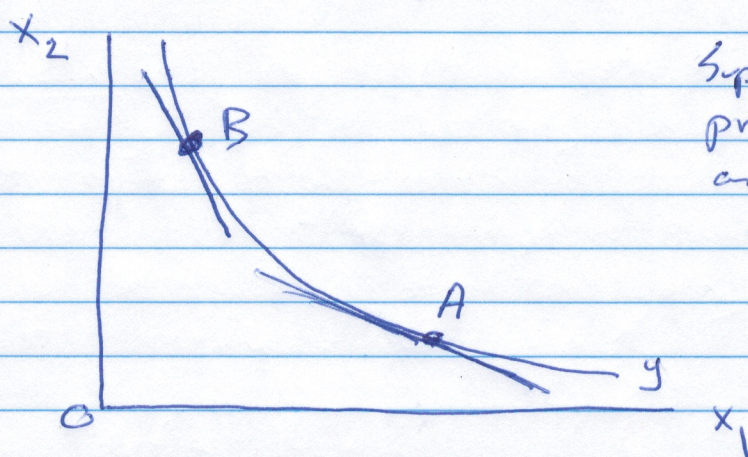
$$\sigma = - \underbrace{\frac{\frac{\partial \frac{x_1(w,y)}{x_2(w,y)}}{\frac{\partial (\frac{w_1}{w_2})}}}_{\text{"slope"}} \cdot \underbrace{\frac{\frac{w_1}{w_2}}{\frac{x_1(w,y)}{x_2(w,y)}}}_{\text{conversion to an elasticity}}$$

This is the definition for the case of two inputs. If there are more inputs, we have to hold the quantities of the other inputs fixed, or hold their prices fixed (it can be done either way, it is just a matter of which method you prefer).

(11)

The first thing to notice about my definition of σ on p. 10 is that it differs from the definition I used in Chapter 1. There I used the quantity ratio $\frac{x_2}{x_1}$ but here I am using $\frac{x_1}{x_2}$. I like this definition better.

- (1) You might be concerned about whether $\frac{x_1(w, y)}{x_2(w, y)}$ is really a function of $\frac{w_1}{w_2}$. Good question! Here is the interpretation. Consider a graph.



Suppose the firm is producing output y and we have strictly convex $V(y)$ sets.

Consider an isocost line described by $C = w_1 x_1 + w_2 x_2$ where total cost C is held constant. The slope of the isocost line is $-\frac{w_1}{w_2}$ (see p. 5)

When $\frac{w_1}{w_2}$ is small, the isocost lines are relatively flat, and the lowest possible isocost line is reached at a point like A. But when $\frac{w_1}{w_2}$ is large, the isocost lines are steeper, and the lowest possible isocost line is reached at a point like B.

Because the slopes of the isocost lines only depend on the ratio $\frac{w_1}{w_2}$, it is this ratio that determines where along the isoquant the solution is located.

For this reason, if we multiply both prices by the same positive scalar t , the input quantities don't change. Thus

$$\begin{aligned} x_1(tw_1, tw_2, y) &= x_1(w_1, w_2, y) \quad \text{for all } t > 0 \\ x_2(tw_1, tw_2, y) &= x_2(w_1, w_2, y) \quad \text{for all } t > 0 \end{aligned}$$

This means that the conditional input demands are homogeneous of degree zero in prices.

Now choose $t = \frac{1}{w_2}$. This gives

$$x_1\left(\frac{w_1}{w_2}, 1, y\right) = x_1(w_1, w_2, y)$$

Likewise, choosing $t = \frac{1}{w_2}$ gives

$$x_2\left(\frac{w_1}{w_2}, 1, y\right) = x_2(w_1, w_2, y)$$

So the ratio $\frac{x_1(w, y)}{x_2(w, y)} = \frac{x_1\left(\frac{w_1}{w_2}, 1, y\right)}{x_2\left(\frac{w_1}{w_2}, 1, y\right)}$

really is a function of $\frac{w_1}{w_2}$.

(2) Note that I defined σ using a negative sign in front of $\frac{\partial \frac{x_1}{x_2}}{\partial \frac{w_1}{w_2}}$. The reason is that this

derivative will normally be negative (as $\frac{w_1}{w_2}$ rises, the ratio $\frac{x_1}{x_2}$ falls - see the graph on p. 11) and I want to define σ so that it is a positive number in what follows.

(13)

One simple way to define a CES production function is

$$f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

To make sure the form of the function is accurate, we should compute σ and verify that it is constant (i.e. σ does not depend on w or y)

Let's consider the Lagrangian $w \cdot x - d[f(x) - y]$

where

$$\text{FOC: } w_1 = d \frac{\partial f}{\partial x_1} = d \left(\frac{1}{\rho} \right) [x_1^\rho + x_2^\rho]^{\frac{1}{\rho} - 1} \rho x_1^{\rho-1}$$

$$w_2 = d \frac{\partial f}{\partial x_2} = d \left(\frac{1}{\rho} \right) [x_1^\rho + x_2^\rho]^{\frac{1}{\rho} - 1} \rho x_2^{\rho-1}$$

Computational note: we don't need to solve for $x_1(w, y)$ and $x_2(w, y)$ separately. This would be hard because it would also require us to solve for $d(w, y)$, so we would have three equations and three unknowns. All we need is a solution for $\frac{x_1}{x_2}$. To get this, divide the top equation by the bottom equation:

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2} \right)^{\rho-1} \quad \text{or} \quad \frac{x_1}{x_2} = \left(\frac{w_1}{w_2} \right)^{\frac{1}{\rho-1}}$$

This trick is often (although not always) useful if you want to solve for input ratios.

$$\text{Now differentiate: } \frac{d\left(\frac{x_1}{x_2}\right)}{d\left(\frac{w_1}{w_2}\right)} = \left(\frac{1}{\rho-1} \right) \left(\frac{w_1}{w_2} \right)^{\frac{1}{\rho-1} - 1}$$

Multiply by $\frac{w_1}{w_2}$ and divide by $\frac{x_1}{x_2} = \left(\frac{w_1}{w_2} \right)^{\frac{1}{\rho-1}}$.

This gives $\boxed{\sigma = \frac{1}{1-\rho}}$ which is a constant.

From the definition of σ and the curvature of the isoquant, we must have $\sigma > 0$, so we must have $\rho < 1$.

[If $\sigma < 0$, we would have isoquants bending the wrong way, and the FOC would give cost max, not cost min.]

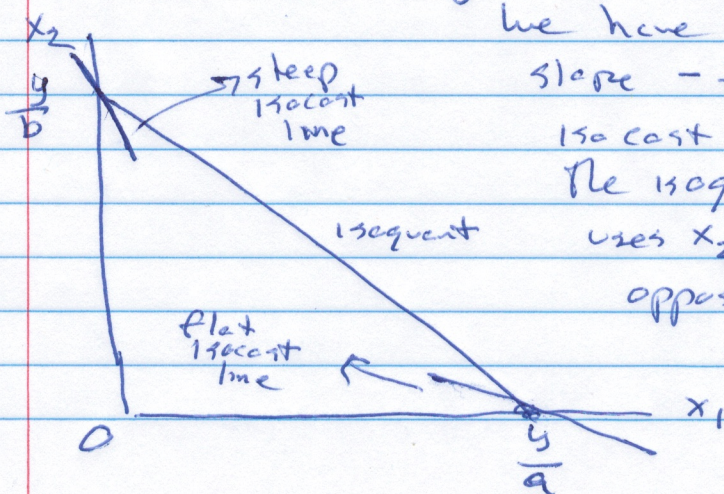
Interpretation.

- ① if σ is "large" ($\rho \rightarrow +1$), then input use is very sensitive to the price ratio (quantity effects dominate)
- ② if σ is "small" (close to zero, so $\rho \rightarrow -\infty$) then input use doesn't respond much to the price ratio (price effects dominate)

Next I will consider some special cases. I will write the CES function as $y = (ax_1^\rho + bx_2^\rho)^{1/\rho}$ where $a > 0$, $b > 0$ to make it slightly more general. (Notice that this implies constant returns to scale)

① Linear production function. Let $\rho \rightarrow +1$ so $\sigma \rightarrow +\infty$

The CES becomes $y = ax_1 + bx_2$.



We have a linear isoquant with slope $-\frac{a}{b}$. If $\frac{w_1}{w_2} > \frac{a}{b}$, the isocost lines are steeper than the isoquant and the firm only uses x_2 . If $\frac{w_1}{w_2} < \frac{a}{b}$, the opposite is true, and the firm only uses x_1 .

Technically, σ is not defined at $\rho = +1$. But intuitively, you can think about this as an extreme case where $\frac{x_1}{x_2}$ is very sensitive to prices. To see why, imagine that we happen to have $\frac{w_1}{w_2} = \frac{a}{b}$ so every point along the isoquant is on the same isocost line (all of these points solve the cost min problem). Suppose we are at some interior point. Any slight change in $\frac{w_1}{w_2}$ will cause the firm to jump to a boundary solution where it uses only one input. In this sense, the elasticity of substitution is extremely high (quantity responses are very big)

Note that with a linear production function, to set up the FOCs correctly, you would need Kuhn-Tucker multipliers, because the non-negativity constraints on the input quantities are important.

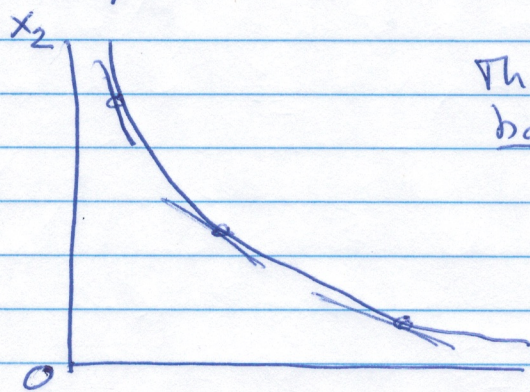
② Cobb-Douglas production function Let $\rho = 0$,
 $\sigma = 1$.

There is a problem here, because with $y = (ax_1^\rho + bx_2^\rho)^{1/\rho}$ and $\rho = 0$ we have an undefined exponent.

However, let $a + b = 1$ and consider $y = x_1^a x_2^b$ which is clearly a Cobb-Douglas function (with CRS).

If you go back to the FOC for the CES, you should be able to show that they are equivalent to the FOCs for this version of Cobb-Douglas. Therefore the firm's behavior is the same in the two cases.

The isoquant looks like this:

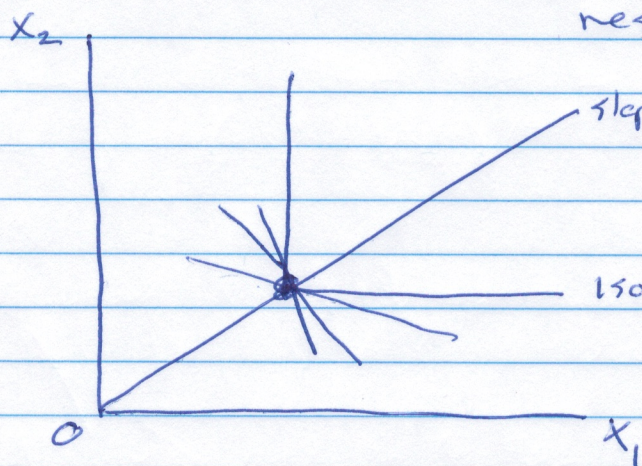


This is the only function that is both CES and CRS.
(with $\sigma = +1$)

The case $\sigma = 1$ is important because it is the dividing line between "elastic" ($\sigma > 1$) and "inelastic" ($\sigma < 1$).

③ Leontief production function Let $\rho \rightarrow -\infty$
so $\sigma \rightarrow 0$.

If we write the CES as $y = [(ax_1)^\rho + (bx_2)^\rho]^{1/\rho}$ which is equivalent to the previous version with a slightly different way of defining a and b , it can be shown that in the limit as $\rho \rightarrow -\infty$ we get $y = \min \{ax_1, bx_2\}$. This is the Leontief case (I will not prove the limit result here.)



In this case, the input ratio $\frac{x_1}{x_2}$ is not affected by the price ratio $\frac{w_1}{w_2}$ (no matter what slope the isocost lines have, the solution is the same).

This is the extreme case where price effects dominate (there is zero quantity response).

The last topic I will discuss is the share of an input in total cost.

Assume we have a CES production function, and it is CRS.

Define $\hat{w} = \frac{w_1}{w_2}$ and $\hat{x} = \frac{x_1}{x_2}$.

Think of input 1 as labor and input 2 as capital.

You might be interested in the distribution of income between labor and capital. You can compute the share of labor in the firm's total cost as

$$\frac{w_1 x_1}{w_1 x_1 + w_2 x_2}. \text{ If we write } s = \frac{w_1 x_1}{w_2 x_2} = \hat{w} \hat{x}, \text{ this}$$

$$\longrightarrow \text{share becomes } \frac{1}{1 + 1/s}$$

This moves in the same direction as s (if $s \uparrow$ then labor's share in total cost $\uparrow$).

Hold output constant (no scale effects). If the wage w_1 goes up, does labor account for a larger or smaller fraction of total cost? We have

$$\frac{\partial s}{\partial \hat{w}} = \hat{x} + \hat{w} \frac{\partial \hat{x}}{\partial \hat{w}} = \hat{x} \left[1 + \frac{\hat{w}}{\hat{x}} \frac{\partial \hat{x}}{\partial \hat{w}} \right] = \frac{x_1}{x_2} (1 - \sigma)$$

Using the definition of σ .

So if $\sigma = +1$ (which is the Cobb-Douglas case) there is no effect, labor's share is unchanged. The reason is that the price effect and the quantity response exactly cancel out.

If $\sigma < +1$ (the "inelastic" case) we have $\frac{\partial s}{\partial w} > 0$, which means s rises and therefore labor's share of total cost also rises.

In this situation, quantity responses are small, price effects dominate, and the higher wage gives a higher labor share. The extreme case would be the Leontief function where the quantity response is zero.

If $\sigma > +1$ (the "elastic" case) we have $\frac{\partial s}{\partial w} < 0$, which means s falls so labor's share also falls.

quantity
effects
dominate

For this situation the firm's quantity responses are proportionately larger than the price effect (there are large substitutions away from inputs that become more expensive). The extreme case would be the linear production function, where we can get discrete jumps from an interior point to a boundary point.

Implication: if you want to study the effect of relative prices on cost shares don't use the Cobb-Douglas function, because you will be assuming no effect. You need a more general production function so you can estimate ρ or σ .

Footnote: in chapter 1, Varian replaced $\frac{L_1}{L_2}$ by $\frac{f_1}{f_2}$ when he defined σ . This is OK as long as cost minimization and price-taking behavior in the input markets can be assumed. It converts the definition of σ into a purely technological concept.